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CANONICAL METRICS IN A CONFORMAL CLASS

1. Introduction and main results.

It is an old and difficult question in differential geometry if there exists a "best" or canonical metric on a smooth manifold, which makes a manifold "most symmetric". Standard examples are round spheres and flat tori, where the word "best" means constant curvature. If there are no assumptions made about a manifold then there is a high chance that there are no reasonable "best" metrics, which was informally explained by Gromov in [2]. Usually one assumes the existence of some geometric or algebraic structure on the manifold and considers only a class of metrics compatible with the structure.

Classical examples include Kahler metrics on compact complex manifolds, left- or biinvariant metrics on Lie groups or metrics compatible with a conformal structure. And canonical metric should ideally be uniquely defined by the structure.

One of the first well-known examples of canonical metrics is of course a hyperbolic metric on a compact Riemann surface of genus $g \geq 2$. In case of Kahler manifolds the Yau's proof of Calabi conjecture provides the existence of a distinguished metric in the same Kahler class as the initial one [6].

For a compact smooth manifold by the solution to the Yamabe problem, achieved in works of Trudinger [5], Aubin [1] and Schoen [4], each conformal structure on a compact manifold supports a metric of a constant scalar curvature. However this metric is in general not unique in the case of positive scalar curvature.

In a more recent work Habermann and Jost [3] construct canonical metrics in a conformal class using Green function of the Yamabe operator. Their construction requires local conformal flatness of a class if the dimension of the manifold is greater than 3.

In this paper we construct canonical metrics in a given conformal class for a $2n$ -dimensional oriented compact smooth manifold M , with non-trivial n -th de Rham cohomology and some natural non-degeneracy assumption on the conformal class.

We use Hodge theory of harmonic forms and the key point, which makes the construction very explicit is a well-known observation that n -dimensional harmonic forms of a $2n$ -dimensional manifold remain harmonic under conformal change of a metric.

We define a functional E on the space of all Riemannian metrics invariant under the natural action of the group $Diff$, which we call a Harmonic Energy. Informally speaking the functional E measures the failure of a wedge product of two harmonic forms to be harmonic. Then we prove that inside a given conformal class there exists a unique normalized metric minimizing E .

Moreover we obtain an explicit formula for the extremal metric in terms of the initial metric representing the conformal class and an orthonormal basis of harmonic n -forms. We also explicitly compute the critical Harmonic Energy and observe that corresponding value can be defined for any conformal class without any non-degeneracy assumptions and so may serve as a conformal invariant of any closed oriented smooth $2n$ -manifold.

In the next chapter we apply our construction to Riemann surfaces, thus producing a natural family of metrics on them.

From now and further by manifold we mean a closed oriented smooth manifold.

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1.1. Hodge theory.

Here we briefly discuss the basics of the Hodge theory and prove a lemma which will be in use later.

Let V be an oriented d -dimensional Euclidean space. Then the inner product produces a natural isomorphism $V = V^*$, which naturally extends to the graded isomorphism of Grassman algebras $\bigwedge^k V = \bigwedge^k V^*$. The latter isomorphism provides a geometric interpretation of the space of k -forms $\Omega^k(V)$. Namely, for any oriented k -plane P , generated by the ordered set of orthonormal vectors e_1, \dots, e_k we define a so-called decomposable k -form ω_P as follows. For a set of vectors v_1, \dots, v_k , the value $\omega_P(v_1, \dots, v_k)$ equals to the algebraic k -volume of the projection of the parallelepiped $\langle v_1, \dots, v_k \rangle$ onto P . In terms of the isomorphism above one can check the identity $\omega_P = e_1 \wedge \dots \wedge e_k$. Any other k -form ω is a linear combination of decomposable k -forms.

The inner product of two decomposable k -forms ω_P, η_Q is defined

as a Jacobian of the orthogonal projection of oriented k -plane P into oriented k -plane Q and then extended by linearity to $\Omega^k(V)$.

The Hodge Star operator $*$: $\Omega^k \rightarrow \Omega^{d-k}$ is also defined in terms of decomposable forms and then extended by linearity on $\Omega^k(V)$. For any positively oriented orthonormal basis e_1, \dots, e_d we put by definition $*(e_1 \wedge \dots \wedge e_k) = e_{k+1} \wedge \dots \wedge e_d$.

Now we consider a d -dimensional Riemannian manifold (M, g) and naturally define the Hodge Star operator $\Omega^k(M) \rightarrow \Omega^{d-k}(M)$ as the Riemannian metric g defines inner product on the tangent space $T_x M$ at each point $x \in M$.

We then define the L^2 - inner product of two k -forms $\omega, \eta \in \Omega^k(M)$ as $(\omega, \eta)_{L^2} = \int (\omega, \eta)_x Vol_g$ where Vol_g is a Riemannian volume form associated to g and we have a well-known equality $(\omega, \eta)_{L^2} = \int \omega \wedge *\eta$.

For a differential $d : \Omega^{k-1} \rightarrow \Omega^k$ there exists a formal adjoint with respect to the inner product defined above and given by the formula $\delta : \Omega^k \rightarrow \Omega^{k-1}$, $\delta = (-1)^k *^{-1} d*$.

The Hodge Laplacian is defined as

$$\Delta : \Omega^k(M) \rightarrow \Omega^k(M)$$

$$\Delta = d\delta + \delta d$$

The form ω is called harmonic if $\Delta\omega = 0$ and it is easy to check that $\Delta\omega = 0$ iff $d\omega = 0, d*\omega = 0$.

Let $\mathcal{H}^k(M)$ be the space of harmonic k -forms. Since all harmonic forms are closed we have a natural map $\Phi : \mathcal{H}^k(M) \rightarrow H^k(M)$, where $H^k(M)$ is a de Rham cohomology of M and a celebrated theorem of Hodge asserts that Φ is an isomorphism.

Lemma 1.1. Let V be a $2n$ -dimensional oriented space with an inner product g and let $*_g : \Omega^n(V) \rightarrow \Omega^n(V)$ be a corresponding Hodge Star operator acting on n -forms. Let also k be a positive constant. Then $*_g = *_k g$.

Proof. Obvious from the definition of the Hodge Star. **Q.E.D.**

Corollary 1.2. Let M be a $2n$ -dimensional manifold, g be a Riemannian metric, $\rho \in C^\infty(M), \rho > 0$ and ω, η be a pair of harmonic n -forms with respect to g . Then ω, η are harmonic with respect to ρg and $(\omega, \eta)_{L^2}^g = (\omega, \eta)_{L^2}^{\rho g}$.

Proof. The first statement follows from the Lemma 1.1. and the fact that harmonic forms ω satisfy $d\omega = 0, d*\omega = 0$.

For the second statement the harmonicity of forms is not essential and it follows from the Lemma 1.1. and a formula mentioned before

$$(\omega, \eta)_{L^2} = \int \omega \wedge * \eta. \quad \mathbf{Q.E.D.}$$

2. Harmonic Energy.

Let M be a $2n$ -dimensional manifold and g be a Riemannian metric on it. Then all spaces $\Omega^k(M)$ inherit an L^2 -inner product defined above and the n -th cohomology space $H^n(M)$ becomes a Euclidean space via the Hodge isomorphism Φ . Then $H^n(M) \otimes H^n(M)$ naturally becomes a Euclidean space and if $e_1 \cdots, e_p$ is an orthonormal basis for $H^n(M)$ then $e_i \otimes e_j, 1 \leq i, j \leq p$ is an orthonormal basis for $H^n(M) \otimes H^n(M)$.

For cohomology classes $[\omega], [\eta]$ and $[\omega \wedge \eta]$ let $\omega_H, \eta_H, (\omega \wedge \eta)_H$ be the corresponding harmonic representatives and let $\overline{\Omega^{2n}(M)}$ be a completion of $\Omega^{2n}(M)$ with respect to the L^2 -norm. Then a linear operator $A : H^n(M) \otimes H^n(M) \rightarrow \overline{\Omega^{2n}(M)}$ is defined by:

$$A([\omega] \otimes [\eta]) = \omega_H \wedge \eta_H - (\omega \wedge \eta)_H.$$

A measures the failure of a wedge product of two harmonic forms to be harmonic. It is a linear operator from Euclidean to a Hilbert space and has a natural norm $|A|$, given by the formula $|A|^2 = \text{Tr}(A^*A)$. Any choice of orthonormal basis $e_1, \dots, e_p \in H^n(M)$ allows us to write an explicit formula:

$$|A|^2 = \sum A(e_i \otimes e_j) \cdot A(e_i \otimes e_j)$$

where $\omega \cdot \eta$ denotes the L^2 -product of $2n$ -forms ω, η inside $\overline{\Omega^{2n}(M)}$

Definition 1. Let g be a Riemannian metric on M . Then its Harmonic Energy is

$$E(g) = |A|^2$$

Definition 2. Let g be a metric on M^{2n} , such that $\int \text{Vol}_g = 1$. Then its Normalized Conformal Class is by definition:

$$C(g) = \{\rho g \mid \int \rho^n \text{Vol}_g = 1, \rho \in C^\infty(M), \rho > 0\}$$

which is just a set of conformally equivalent metrics with total volume equal to one.

MAIN THEOREM. Let (M^{2n}, g_0) be a Riemannian manifold such that for any $x \in M$ there exists a form $\omega \in \mathcal{H}^n(M), \omega(x) \neq 0$.

Then there exists a unique metric $g \in C(g_0)$ minimizing E .

Remark. The assumption of the theorem of course immediately implies $H^n(M) \neq 0$.

PROOF.

Let us pick a set of harmonic forms ξ_1, \dots, ξ_p which form an orthonormal basis in $H^n(M)$ for any metric from $C(g_0)$. This is a crucial ingredient of the proof and such a choice is possible because of the Corollary 1.2.

Now we consider any metric $g = \rho g_0 \in C(g_0)$ and calculate $E(g)$ using that g -harmonic $2n$ -forms are proportional to the Vol_g :

$$E(g) = \sum A(\xi_i \otimes \xi_j) \cdot A(\xi_i \otimes \xi_j) = \sum \int A(\xi_i \otimes \xi_j) \wedge *A(\xi_i \otimes \xi_j) = \sum \int (\xi_i \wedge \xi_j - (\int \xi_i \wedge \xi_j) Vol_g) \wedge *(\xi_i \wedge \xi_j - (\int \xi_i \wedge \xi_j) Vol_g)$$

Let us now introduce the following notations: let f_{ij} be smooth functions, defined by $\xi_i \wedge \xi_j = f_{ij} Vol_0$ and c_{ij} be constants, defined by $c_{ij} = \int \xi_i \wedge \xi_j$.

Using that $Vol_g = \rho^n g_0$, we can rewrite

$$E(g) = \sum \int (f_{ij} \rho^{-n} - c_{ij})^2 \rho^n Vol_0. \text{ Opening brackets we obtain}$$

$$E(g) = \int f^2 \rho^{-n} Vol_0 - C^2 \text{ where } f = \sqrt{\sum f_{ij}^2} \text{ and } C = \sqrt{\sum c_{ij}^2}.$$

Now let us prove by contradiction that $f(x) > 0$ for any $x \in M$. Indeed $f(x) = 0 \implies f_{ij}(x) = 0 \implies \omega \wedge \eta(x) = 0$ for any $\omega, \eta \in \mathcal{H}^n(M)$ as i, j run through the basis of $\mathcal{H}^n(M)$. But then for any $\omega \in \mathcal{H}^n(M)$, $(\omega, \omega)_x Vol = \omega \wedge * \omega(x) = 0$ which implies $\omega(x) = 0$ and contradicts the theorem assumption.

To find a minimum of $E(g)$ among all metrics $g \in C(g_0)$ we use a form of integral Cauchy inequality on the functions f, ρ :

$$\int \frac{f^2}{\rho^n} Vol_0 \cdot \int \rho^n Vol_0 \geq \left(\int f Vol_0 \right)^2$$

which immediately implies that the extremal metric is $g = \rho g_0$ with $\rho = (f / \int f Vol_0)^{1/n}$ and minimal $E(g) = (\int f Vol_0)^2 - C^2$. **Q.E.D.**

Remark 1. Using explicit formulas from the proof one can easily check that the minimizing metric and critical harmonic energy are indeed independent on the initial metric g_0 we started with.

Remark 2. As the expression for the critical energy $E(g)$ is independent on the initial metric g_0 one can define the harmonic energy of a given conformal class on any compact smooth oriented manifold M^{2n} dropping the assumption of the theorem, which is only required to guarantee the non-degeneracy of the critical metric.

3. Canonical metrics on Riemann surfaces.

Consider a closed oriented surface M of genus $g \geq 2$. The conformal classes of metrics on M are in natural one-to-one correspondence with complex structures on it. Let us fix a conformal class on M . Then harmonic 1-forms for such a conformal class are precisely the real parts of abelian differentials for the corresponding complex structure.

As it is well known that for any $x \in M$ there exists an abelian differential $w, w(x) \neq 0$ we have that either $Re(w) \neq 0$ or $Im(w) = Re(-iw) \neq 0$ which means that natural conformal class of any Riemann surface satisfies the assumptions of the main theorem.

So we produced a canonical metric for any Riemann surface.

4. Open questions.

1. It seems to be an interesting question to describe the properties of the metric on the Riemann surface, defined above, and in particular to see if its curvature is negative on M .

2. A manifold is called formal if there exists a metric such that the wedge product of any two harmonic forms is harmonic. It is not hard to show that any closed surface of genus $g \geq 2$ is not formal which implies that the value of critical Harmonic Energy is a positive smooth function E_g on the moduli space \mathcal{M}_g of complex curves of genus g .

As \mathcal{M}_g is not compact it is an interesting question if E_g is strictly positive on \mathcal{M}_g which can be reformulated as if M is formal "at infinity" of the moduli space.

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